5.6 Forwards and Futures

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5.6. 1 Forward Contracts

Let $S(t)$, $0 \le t \le \overline{T}$, be an asset price process, and let $R(t)$, $0 \le t \le \overline{T}$, be an interest rate process. We choose here some large time $\overline{T}$, and all bonds and derivative securities we consider will mature or expire at or before time $\overline{T}$. As usual, we define the discount process $D(t) = e^{-\int_0^t R(u)du}$. According to the risk-neutral pricing formula (5.2.30), the price at time t of a zero-coupon bond paying 1 at time T is

$$B(t,T) = \frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)], \quad 0 \le t \le T \le \overline{T}. \tag{5.6.1}$$

$$D(t)V(t) = \widetilde{\mathbb{E}}\big[D(T)\underbrace{V(T)}_{=1}|\mathcal{F}(t)\big], \; 0 \le t \le T. \tag{5.2.30}$$

$V(T)$ represents the payoff at time T of a derivative security.

Definition 5.6.1

- A forward contract is an agreement to pay a specified delivery price K at a delivery date T, where $0 \le T \le \overline{T}$ for the asset whose price at time t is S(t). The T-forward price $For_S(t,T)$ of this asset at time t where $0 \le t \le T \le \overline{T}$ is the value of K that makes the forward contract have no-arbitrage price zero at time t.

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的遠期契約

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Theorem 5.6.2.

下一頁證明

- Assume that zero-coupon bonds of all maturities can be traded. Then

$$\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}, \quad 0 \le t \le T \le \overline{T}. \tag{5.6.2}$$

PROOF: Suppose that at time t an agent sells the forward contract with delivery date T and delivery price K. Suppose further that the value K is chosen so that the forward contract has price zero at time t. Then selling the forward contract generates no income. Having sold the forward contract at time t, suppose the agent immediately shorts $\frac{S(t)}{B(t,T)}$ zero-coupon bonds and uses the income $S(t)$ generated to buy one share of the asset. The agent then does no further trading until time T, at which time she owns one share of the asset, which she delivers according to the forward contract. In exchange, she receives K. After covering the short bond position, she is left with $K - \frac{S(t)}{B(t,T)}$. If this is positive, the agent has found an arbitrage.

In order to preclude arbitrage, K must be given by (5.6.2).

$$\boxed{\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}, \quad 0 \le t \le T \le \overline{T}. \qquad (5.6.2)}$$

Remark 5. 6. 3.

- The forward price must be given by (5.6.2) in order to preclude arbitrage. Because we have assumed the existence of a risk-neutral measure and are pricing all assets by the risk-neutral pricing formula,we must be able to obtain (5.6.2) from the risk-neutral pricing formula as well. We compute the price at time t of the forward contract to be

risk-neutral pricing formula

$$D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]$$

$$\begin{aligned}
&\frac{1}{D(t)}\widetilde{\mathbb{E}}\big[D(T)\big(S(T)-K\big)\big|\mathcal{F}(t)\big] \\
&= \frac{1}{D(t)}\widetilde{\mathbb{E}}\big[D(T)S(T)\big|\mathcal{F}(t)\big] - \frac{K}{D(t)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)] \\
&= S(t) - KB(t,T).
\end{aligned}$$

$$B(t,T) = \frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)]$$

$$\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}$$

=0

In order for this to be zero, K must be given by (5.6.2).

5.6.2 Futures Contracts

- Consider a time interval [0,T], which we divide into subintervals using the partition points 0= $t_0 < t_1 < \ldots < t_n = T$ We shall refer to each subinterval $[t_k, t_{k+1})$ as a "day."
- suppose the interest rate is constant within each day. Then the discount process is given by D(0)=1 and, for k=0,1,…,n-1, $\boxed{D(t) = e^{-\int_0^t R(u)du}}$

$$D(t_{k+1}) = \exp\{-\int_0^{t_{k+1}} R(u)du\} = \exp\{-\sum_{j=0}^{K} R(t_j)(t_{j+1} - t_j)\},$$

which is $F(t_k)$-measurable.

According to the risk-neutral pricing formula (5.6.1), the zero-coupon bond paying 1 at maturity T has time- t_k price

$$B(t_k, T) = \frac{1}{D(t_k)} \widetilde{\mathbb{E}}[D(T) | \mathcal{F}(t_k)].$$

$$\boxed{B(t,T) = \frac{1}{D(t)} \widetilde{\mathbb{E}}[D(T) | \mathcal{F}(t)]}$$

An asset whose price at time t is S(t) has time- t_k forward price

$$\boxed{\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}}$$

$$For_s(t_k, T) = \frac{S(t_k)}{B(t_k, T)}, \qquad (5.6.3)$$

an $F(t_k)$ -measurable quantity.

Suppose we take a long position in the forward contract at time t_k (i.e., agree to receive $S(T)$ and pay $\text{For}_S(t_k, T)$ at time T). The value of this position at time $t_j \geq t_k$ is

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在t_k買遠期契約

$$V_{k,j} = \frac{1}{D(t_j)} \widetilde{\mathbb{E}} \left[D(T) \left(S(T) - \frac{S(t_k)}{B(t_k, T)} \right) \middle| \mathcal{F}(t_j) \right]$$

$$= \frac{1}{D(t_j)} \widetilde{\mathbb{E}}[D(T)S(T) | \mathcal{F}(t_j)] - \frac{S(t_k)}{B(t_k, T)} \cdot \frac{1}{D(t_j)} \widetilde{\mathbb{E}}[D(T) | \mathcal{F}(t_j)]$$

$$= S(t_j) - S(t_k) \cdot \frac{B(t_j, T)}{B(t_k, T)}.$$

$$\boxed{D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T) | \mathcal{F}(t)] \quad \text{For}_S(t, T) = \frac{S(t)}{B(t, T)}}$$

$$\boxed{B(t, T) = \frac{1}{D(t)} \widetilde{\mathbb{E}}[D(T) | \mathcal{F}(t)]}$$

$$\boxed{D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T) | \mathcal{F}(t)]}$$

$$\boxed{\begin{aligned} V_{k,j} &= \frac{1}{D(t_j)}\widetilde{\mathbb{E}}\left[D(T)\left(S(T) - \frac{S(t_k)}{B(t_k,T)}\right)\middle|\mathcal{F}(t_j)\right] \\ &= \frac{1}{D(t_j)}\widetilde{\mathbb{E}}[D(T)S(T)|\mathcal{F}(t_j)] - \frac{S(t_k)}{B(t_k,T)} \cdot \frac{1}{D(t_j)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t_j)] \\ &= S(t_j) - S(t_k) \cdot \frac{B(t_j,T)}{B(t_k,T)}. \end{aligned}}$$

$\boxed{S(t_j) - S(t_j)}$

If $t_j = t_k$, this is zero, as it should be. However, for $t_j > t_k$, it is generally different from zero. For example, if the interest rate is a constant r so that $B(t,T) = e^{-r(T-t)}$, then

$$\boxed{B(t,T) = \underset{e^{-rt}}{\frac{1}{D(t)}}\widetilde{\mathbb{E}}[\underset{e^{-rT}}{D(T)}|\mathcal{F}(t)]}$$

$$V_{k,j} = S(t_j) - \underset{\boxed{e^{-r(T-tj)}Xe^{r(T-tk)}}}{e^{r(t_j - t_k)}}S(t_k).$$

To alleviate the problem of default risk, parties to a forward contract could agree to settle one day after the contract is entered. The original forward contract purchaser could then seek to purchase a new forward contract one day later than the initial purchase. By repeating this process, the forward contract purchaser could generate the cash flow

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$$\boxed{V_{k,j} = S(t_j) - S(t_k) \cdot \frac{B(t_j,T)}{B(t_k,T)}.}$$

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$$V_{0,1} = S(t_1) - S(t_0) \cdot \frac{B(t_1,T)}{B(t_0,T)} = S(t_1) - S(t_0) \cdot \frac{B(t_1,T)}{B(0,T)},$$

$$V_{1,2} = S(t_2) - S(t_1) \cdot \frac{B(t_2,T)}{B(t_1,T)}$$

$$\vdots$$

$$V_{n-1,n} = S(t_n) - S(t_{n-1}) \cdot \frac{\overset{=1}{B(t_n,T)}}{B(t_{n-1},T)} = S(T) - \frac{S(t_{n-1})}{B(t_{n-1},T)}.$$

Problems

- The purchaser of the forward contract was presumably motivated by a desire to hedge against a price increase in the underlying asset. It is not clear the extent to which receiving this cash flow provides such a hedge.
- This daily buying and selling of forward contracts requires that there be a liquid market each day for forward contracts initiated that day and forward contracts initiated one day before.

到期日為T的期貨在t的價格

A better idea than daily repurchase of forward contracts is to create a *futures price* $\mathrm{Fut}_S(t,T)$, and use it as described below. If an agent holds a long futures position between times t_k and t_{k+1}, then at time t_{k+1} he receives a payment

$$\mathrm{Fut}_S(t_{k+1},T) - \mathrm{Fut}_S(t_k,T). \quad \boxed{\textit{marking to margin}}$$

$\mathrm{Fut}_S(t,T)$ is $\mathcal{F}(t)$-measurable for every t and

$$\mathrm{Fut}_S(T,T) = S(T).$$

- The sum of payments received by an agent who purchases a futures contract at time zero and holds it until delivery date T is

$$(\cancel{Fut_S(t_1,T)} - Fut_S(t_0,T)) + (\cancel{Fut_S(t_2,T)} - \cancel{Fut_S(t_1,T)}) + \cdots$$

$$\cdots + (Fut_S(t_n,T) - \cancel{Fut_S(t_{n-1},T)}) = (Fut_S(T,T) - Fut_S(0,T))$$

$$\boxed{\mathrm{Fut}_S(T,T) = S(T)} \quad = S(T) - Fut_S(0,T).$$

at time t_{k+1} he receives $\mathrm{Fut}_S(t_{k+1},T) - \mathrm{Fut}_S(t_k,T)$

If the agent takes delivery of the asset at time T, paying market price S(T) for it. Ignoring the time value of money, he has effectively paid the price Futs(0, T) for the asset, a price that was locked in at time zero.

$$\boxed{S(T) - Fut_S(0,T)}$$

-S(T)+ S(T)- Futs(0, T) =- Futs(0, T)

- In addition to satisfying Futs(T, T) = S(T), the futures price process is chosen so that at each time t_k the value of the payment to be received at time t_{k+l} , and indeed at all future times $t_j > t_k$, is zero.
- This means that at any time one may enter or close out a position in the contract without incurring any cost other than payments already made.

- The condition that the value at time t_k of the payment to be received at time t_{k+l} be zero may be written as

$$\boxed{D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]}$$

$$\begin{aligned} 0 &= \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\Big[D(t_{k+1})\big(\mathrm{Fut}_S(t_{k+1},T) - \mathrm{Fut}_S(t_k,T)\big)\Big|\mathcal{F}(t_k)\Big] \\ &= \frac{D(t_{k+1})}{D(t_k)}\Big\{\widetilde{\mathbb{E}}[\mathrm{Fut}_S(t_{k+1},T)|\mathcal{F}(t_k)] - \mathrm{Fut}_S(t_k,T)\Big\}, \end{aligned}$$

$$\boxed{D(t_{k+1}) \text{ is } \mathcal{F}(t_k)\text{-measurable}}$$

$$\boxed{0 = \frac{D(t_{k+1})}{D(t_k)} \{\widetilde{\mathbb{E}}[\mathrm{Fut}_S(t_{k+1}, T) | \mathcal{F}(t_k)] - \mathrm{Fut}_S(t_k, T)\}}$$

- From the equation above, we see that

$$\widetilde{\mathbb{E}}[\mathrm{Fut}_S(t_{k+1}, T) | \mathcal{F}(t_k)] = \mathrm{Fut}_S(t_k, T), \quad k = 0, 1, \ldots, n-1. \tag{5.6.4}$$

This shows that $\mathrm{Fut}_S(t_k, T)$ must be a discrete-time martingale under $\widetilde{\mathbb{P}}$.

- But we also require that Futs(T, T) = S(T), from which we conclude that the futures prices must be given by the formula

$$\mathrm{Fut}_S(t_k, T) = \widetilde{\mathbb{E}}[S(T) | \mathcal{F}(t_k)], \ k = 0, 1, \ldots, n. \tag{5.6.5}$$

$$\boxed{\widetilde{\mathbb{E}}[S(T) | \mathcal{F}(t_k)] = \widetilde{\mathbb{E}}[\ \mathrm{Fut}_S(T, T) | \mathcal{F}(t_k)] = \mathrm{Fut}_S(t_k, T)}$$

The value at time t_k of the payment to be received at time t_j is zero for every $j \geq k+1$.

$$\boxed{D(t)V(t) = \widetilde{\mathbb{E}}\big[D(T)V(T)\big|\mathcal{F}(t)\big]}$$

$$\frac{1}{D(t_k)}\widetilde{\mathbb{E}}\Big[D(t_j)\big(\mathrm{Fut}_S(t_j,T) - \mathrm{Fut}_S(t_{j-1},T)\big)\Big|\mathcal{F}(t_k)\Big]$$

(Iterated conditioning) $\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]\big|\mathcal{H}\big] = \mathbb{E}[X|\mathcal{H}]$ (2.3.20)

$$= \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\Big[\widetilde{\mathbb{E}}\Big[D(t_j)\big(\mathrm{Fut}_S(t_j,T) - \mathrm{Fut}_S(t_{j-1},T)\big)\Big|\mathcal{F}(t_{j-1})\Big]\Big|\mathcal{F}(t_k)\Big]$$

$$= \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\Big[D(t_j)\widetilde{\mathbb{E}}\big[\mathrm{Fut}_S(t_j,T)\big|\mathcal{F}(t_{j-1})\big] - D(t_j)\mathrm{Fut}_S(t_{j-1},T)\Big|\mathcal{F}(t_k)\Big]$$

$\mathcal{F}(t_{j-1})$-measurability of $D(t_j)$

$$= \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\Big[\cancel{D(t_j)\mathrm{Fut}_S(t_{j-1},T)} - \cancel{D(t_j)\mathrm{Fut}_S(t_{j-1},T)}\Big|\mathcal{F}(t_k)\Big] = 0.$$

the martingale property for $\mathrm{Fut}_S(t,T)$

Definition 5.6.4. *The* futures price *of an asset whose value at time T is $S(T)$ is given by the formula* $\boxed{\mathrm{Fut}_S(T,T) = S(T)}$

$$\mathrm{Fut}_S(t,T) = \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)], \quad 0 \le t \le T. \tag{5.6.6}$$

Theorem 5.6.5. *The futures price is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$, it satisfies $\mathrm{Fut}_S(T,T) = S(T)$, and the value of a long (or a short) futures position to be held over an interval of time is always zero.*

下一頁證明

If the filtration $\mathcal{F}(t)$, $0 \leq t \leq T$, is generated by a Brownian motion $W(t)$, $0 \leq t \leq T$, then Corollary 5.3.2 of the Martingale Representation Theorem implies that

$$\mathrm{Fut}_S(t,T) = \mathrm{Fut}_S(0,T) + \int_0^t \widetilde{\Gamma}(u)\, d\widetilde{W}(u), \quad 0 \leq t \leq T,$$

for some adapted integrand process $\widetilde{\Gamma}$ (i.e., $d\mathrm{Fut}_S(t,T) = \widetilde{\Gamma}(t)\, d\widetilde{W}(t)$).

Corollary 5.3.2.

Now let $\widetilde{M}(t)$, $0 \leq t \leq T$, be a martingale under $\widetilde{\mathbb{P}}$. Then there is an adapted process $\widetilde{\Gamma}(u)$, $0 \leq u \leq T$, such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)\, d\widetilde{W}(u),\ 0 \leq t \leq T. \tag{5.3.2}$$

Let $0 \le t_0 < t_1 \le T$ be given and consider an agent who at times t between times t_0 and t_1 holds $\Delta(t)$ futures contracts. It costs nothing to change the position in futures contracts, but because the futures contracts generate cash flow, the agent may have cash to invest or need to borrow in order to execute this strategy. He does this investing and/or borrowing at the interest rate $R(t)$ prevailing at the time of the investing or borrowing. The agent's profit $X(t)$ from this trading satisfies

Capital gain on the futures position

$$dX(t) = \Delta(t)\, d\mathrm{Fut}_S(t,T) + R(t)X(t)\,dt = \Delta(t)\widetilde{\Gamma}(t)\,d\widetilde{W}(t) + R(t)X(t)\,dt,$$

Interest earings on the cash position

$$\boxed{d\mathrm{Fut}_S(t,T) = \widetilde{\Gamma}(t)\,d\widetilde{W}(t)}$$

and thus

$$d\big(D(t)X(t)\big) = D(t)\Delta(t)\widetilde{\Gamma}(t)\,d\widetilde{W}(t). \qquad (5.2.18)$$

D(t)dX(t)+dD(t)X(t)+dD(t)dX(t)

$=D(t)\,\Delta(t)\widetilde{\Gamma}(t)\,d\widetilde{W}(t) + D(t)\,\cancel{R(t)X(t)\,dt} - D(t)\,\cancel{R(t)X(t)\,dt} + 0$

$$dD(t) = -R(t)D(t)\,dt.$$

$$\boxed{d\big(D(t)X(t)\big) = D(t)\Delta(t)\widetilde{\Gamma}(t)\,d\widetilde{W}(t).}$$

Assume that at time t_0 the agent's profit is $X(t_0) = 0$. At time t_1, the agent's profit $X(t_1)$ will satisfy

$$D(t_1)X(t_1) = \int_{t_0}^{t_1} D(u)\Delta(u)\widetilde{\Gamma}(u)\,d\widetilde{W}(u). \tag{5.6.7}$$

$\boxed{D(t_0)X(t_0)=0}$

$$\boxed{D(t_1)X(t_1) = \int_{t_0}^{t_1} D(u)\Delta(u)\widetilde{\Gamma}(u)\,d\widetilde{W}(u).}$$

Because Itô integrals are martingales, we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}[D(t_1)X(t_1)|\mathcal{F}(t_0)] \\
&= \widetilde{\mathbb{E}}\left[\int_0^{t_1} D(u)\Delta(u)\widetilde{\Gamma}(u)\,d\widetilde{W}(u) - \int_0^{t_0} D(u)\Delta(u)\widetilde{\Gamma}(u)\,d\widetilde{W}(u)\middle|\,\mathcal{F}(t_0)\right] \\
&= \cancel{\widetilde{\mathbb{E}}\left[\int_0^{t_1} D(u)\Delta(u)\widetilde{\Gamma}(u)\,d\widetilde{W}(u)\middle|\,\mathcal{F}(t_0)\right]} - \cancel{\int_0^{t_0} D(u)\Delta(u)\widetilde{\Gamma}(u)\,d\widetilde{W}(u)} \\
&= 0. \qquad (5.6.8)
\end{aligned}$$

martingale

$$\int_0^{t_0} D(u)\Delta(u)\widetilde{\Gamma}(u)\,d\widetilde{W}(u)$$

risk-neutral pricing formula

$$D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]$$

According to the risk-neutral pricing formula, the value at time t_0 of a payment of $X(t_1)$ at time t_1 is $\frac{1}{D(t_0)}\widetilde{\mathbb{E}}\big[D(t_1)X(t_1)\big|\mathcal{F}(t_0)\big]$, and we have just shown that this is zero.

$$\boxed{\widetilde{\mathbb{E}}[D(t_1)X(t_1)|\mathcal{F}(t_0)] = 0.}$$

$$dX(t) = \Delta(t)\, d\mathrm{Fut}_S(t,T) + R(t)X(t)\, dt$$

$d(D(t)X(t)) =$ D(t)dX(t)+dD(t)X(t)+dD(t)dX(t)

$= D(t)\Delta(t)\, d\mathrm{Fut}_S(t,T) + \cancel{D(t)R(t)X(t)\, dt} - \cancel{D(t)R(t)X(t)\, dt} + 0$

$$dD(t) = -R(t)D(t)\, dt \quad (5.2.18)$$

If the filtration $\mathcal{F}(t)$, $0 \le t \le T$, is not generated by a Brownian motion, so that we cannot use Corollary 5.3.2, then we must write (5.6.7) as

$$D(t_1)X(t_1) = \int_{t_0}^{t_1} D(u)\Delta(u)\, d\mathrm{Fut}_S(u,T). \tag{5.6.9}$$

$\boxed{D(t_0)X(t_0)=0}$

$\boxed{\text{martingale}}$ $\widetilde{\mathbb{E}}[D(t_1)X(t_1)|\mathcal{F}(t_0)] = 0.$

$\boxed{D(t_0)X(t_0)=0}$

Forwards

$$\frac{1}{D(t)}\widetilde{\mathbb{E}}\big[D(T)\big(S(T)-K\big)\big|\mathcal{F}(t)\big]$$

$$= \frac{1}{D(t)}\widetilde{\mathbb{E}}\big[D(T)S(T)\big|\mathcal{F}(t)\big] - \frac{K}{D(t)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)]$$

$$= S(t) - KB(t,T).$$

=0

Futures

$$0 = \frac{1}{D(t_k)}\widetilde{\mathbb{E}}\big[D(t_{k+1})\big(\mathrm{Fut}_S(t_{k+1},T) - \mathrm{Fut}_S(t_k,T)\big)\big|\mathcal{F}(t_k)\big]$$

$$= \frac{D(t_{k+1})}{D(t_k)}\big\{\widetilde{\mathbb{E}}[\mathrm{Fut}_S(t_{k+1},T)|\mathcal{F}(t_k)] - \mathrm{Fut}_S(t_k,T)\big\},$$

Remark 5.6.6 (Risk-neutral valuation of a cash flow). Suppose an asset generates a cash flow so that between times 0 and u a total of $C(u)$ is paid, where $C(u)$ is $\mathcal{F}(u)$-measurable. Then a portfolio that begins with one share of this asset at time t and holds this asset between times t and T, investing or borrowing at the interest rate R as necessary, satisfies

Capital gain on the asset position

X(u):profit

$$dX(u) = dC(u) + R(u)X(u)\,du,$$

Interest earings on the cash position

or equivalently

$$d\big(D(u)X(u)\big) = D(u)\,dC(u).$$

D(u)dX(u)+dD(u)X(u)+dD(u)dX(u)

$=D(u)\,dC(u) + D(u)\cancel{R(u)X(u)\,du} - D(u)\cancel{R(u)X(u)\,du} + 0$

$$D(t) = e^{-\int_0^t R(u)du} \qquad dD(t) = -R(t)D(t)\,dt.$$

$$\boxed{d\big(D(u)X(u)\big) = D(u)\,dC(u).}$$

Suppose $X(t) = 0$. Then integration shows that

$$D(T)X(T) = \int_t^T D(u)\,dC(u).$$

$\boxed{D(t)X(t)=0}$

The risk-neutral value at time t of $X(T)$, which is the risk-neutral value at time t of the cash flow received between times t and T, is thus

$$\frac{1}{D(t)}\widetilde{\mathbb{E}}\big[D(T)X(T)\big|\mathcal{F}(t)\big] = \frac{1}{D(t)}\widetilde{\mathbb{E}}\left[\int_t^T D(u)\,dC(u)\,\middle|\,\mathcal{F}(t)\right], \quad 0 \le t \le T. \tag{5.6.10}$$

$$\boxed{D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]}$$

$$\frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)X(T)|\mathcal{F}(t)] = \frac{1}{D(t)}\widetilde{\mathbb{E}}\left[\int_t^T D(u)\,dC(u)\middle|\mathcal{F}(t)\right], \quad 0 \le t \le T. \tag{5.6.10}$$

$$D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]$$

Formula (5.6.10) generalizes the risk-neutral pricing formula (5.2.30) to allow for a cash flow rather than payment at the single time T. In (5.6.10), the process $C(u)$ can represent a succession of lump sum payments $A_1, A_2, \ldots, A_n$ at times $t_1 < t_2 < \cdots < t_n$, where each A_i is an $\mathcal{F}(t_i)$-measurable random variable. The formula for this is

$$C(u) = \sum_{i=1}^{n} A_i \mathbb{I}_{[0,u]}(t_i).$$

$$\boxed{C(u) = \sum_{i=1}^{n} A_i \mathbb{I}_{[0,u]}(t_i).}$$

$$\int_t^T D(u)\, dC(u) = \sum_{i=1}^{n} D(t_i) A_i \mathbb{I}_{(t,T]}(t_i).$$

Only payments made strictly later than time t appear in this sum. Equation (5.6.10) says that the value at time t of the string of payments to be made strictly later than time t is

$$\frac{1}{D(t)} \widetilde{\mathbb{E}} \left[\sum_{i=1}^{n} D(t_i) A_i \mathbb{I}_{(t,T]}(t_i) \middle| \mathcal{F}(t) \right] = \sum_{i=1}^{n} \mathbb{I}_{(t,T]}(t_i) \frac{1}{D(t)} \widetilde{\mathbb{E}}[D(t_i) A_i | \mathcal{F}(t)],$$

which is the sum of the time-t values of the payments made strictly later than time t.

$$\boxed{\frac{1}{D(t)} \widetilde{\mathbb{E}}[D(T)X(T) | \mathcal{F}(t)] = \frac{1}{D(t)} \widetilde{\mathbb{E}} \left[\int_t^T D(u)\, dC(u) \middle| \mathcal{F}(t) \right], \quad 0 \le t \le T. \qquad (5.6.10)}$$

5.6.3 Forward–Futures Spread

$$\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)},$$

$$\mathrm{Fut}_S(t,T) = \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)].$$

$$B(t,T) = \frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)|\mathcal{F}(t)]$$

e^{-rt} e^{-rT}

If the interest rate is a constant r, then $B(t,T) = e^{-r(T-t)}$ and

$$\boxed{D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]}$$

$$\mathrm{For}_S(t,T) = e^{r(T-t)}S(t),$$

$$\mathrm{Fut}_S(t,T) = e^{rT}\widetilde{\mathbb{E}}\big[\underbrace{e^{-rT}}_{D(T)}S(T)\big|\mathcal{F}(t)\big] = e^{rT}\underbrace{e^{-rt}}_{D(t)}S(t) = e^{r(T-t)}S(t).$$

In this case, the forward and futures prices agree.

We compare $\mathrm{For}_S(0,T)$ and $\mathrm{Fut}_S(0,T)$ in the case of a random interest rate. In this case, $B(0,T) = \widetilde{\mathbb{E}}D(T)$, and the so-called *forward–futures spread* is

$$\boxed{\begin{aligned}\mathrm{For}_S(t,T) &= \frac{S(t)}{B(t,T)},\\ \mathrm{Fut}_S(t,T) &= \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)].\end{aligned}}$$

$$\begin{aligned}
\mathrm{For}_S(0,T) - \mathrm{Fut}_S(0,T) &= \frac{S(0)}{\widetilde{\mathbb{E}}D(T)} - \widetilde{\mathbb{E}}S(T) \\
&= \frac{1}{\widetilde{\mathbb{E}}D(T)}\left\{\widetilde{\mathbb{E}}[D(T)S(T)] - \widetilde{\mathbb{E}}D(T) \cdot \widetilde{\mathbb{E}}S(T)\right\} \\
&= \frac{1}{B(0,T)}\widetilde{\mathrm{Cov}}\big(D(T), S(T)\big), \qquad (5.6.11)
\end{aligned}$$

where $\widetilde{\mathrm{Cov}}\big(D(T), S(T)\big)$ denotes the covariance of $D(T)$ and $S(T)$ under the risk-neutral measure. If the interest rate is nonrandom, this covariance is zero and the futures price agrees with the forward price.